Chapter Summary

Kinematics

A rigid body undergoing any motion other than translation has an instantaneous axis of rotation. The direction of this axis at a particular instant and the rate at which the rigid body rotates about the axis are specified by its angulat velocity vector ω .

The velocity of a point A of a rigid body is given in terms of the angular velocity vector and the velocity of a point B by

$$V_A = V_B + \omega \times r_{A/B} \qquad \text{Eq.(9.1)}$$

The acceleration of a point A of a rigid body is given in terms of the angular velocity vector, the angular acceleration vector $\alpha = d\omega/dt$,and the acceleration of a point B by

$$a_A = a_B + \alpha \times r_{A/B} + \omega \times (\omega \times r_{A/B}) \qquad \text{Eq.(9.2)}$$

Let Ω be the angular velocity of a rotating coordinate system *xyz* relatuve to a fixed reference frame, and let ω_{rel} be the angular velocity of a rigid body *relative to* the *xyz* system. The rigid body's angular velocity and angular acceleration relative to the fixed reference frame are

$$\begin{gathered} \omega = \Omega + \omega_{rel} \\ \alpha = \frac{d\Omega}{dt} + \frac{d\omega_{relx}}{dt} + \frac{d\omega_{rely}}{dt} + \frac{d\omega_{relz}}{dt} + \Omega \times \omega_{rel} \end{gathered} \qquad \text{Eq.(9.3)}$$

Angular Momentum

If a rigid body rotates about a fixed point ***O*** with angular velocity ω (Fig. a), the components of its angular momentum about ***O*** are given by

$$\begin{bmatrix} H_{ox} \\ H_{oy} \\ H_{oz} \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

This equation also gives the components of the rigid body's angular momentum about the center of mass in general three-dimensional motion (Fig. b).
In that case the moments of inertia are evaluated in terms of a coordinate system with its origin at the cen ter of mass.

Euler Equations

The equations governing three-dimensional motion of a rigid body include Newton's second law and equations of angular motion. For a rigid body rotating about a fixed point ***O*** (Fig. a), the equations of angularmotion are expressed in terms of the components of the total moment about ***O***:

$$\begin{bmatrix} \sum M_{ox} \\ \sum M_{oy} \\ \sum M_{oz} \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where Ω is the angular velocity of the coordinate system. If the coordinate system is body-fixed, $\Omega = \omega$. In the case of general three-dimensional motion (Fig. b), the equation of angular motion are identical except that they are expressed in terms of the components of the total moment about the center of mass.

The rigid body's angular acceleration is related to the derivatives of the components of ω by

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega_x}{dt}\mathbf{i} + \frac{d\omega_y}{dt}\mathbf{j} + \frac{d\omega_z}{dt}\mathbf{k} + \Omega \times \omega$$

If the coordinate system does not rotate or is body-fixed, the terms $d\omega_x/dt$,$d\omega_y/dt$, and $d\omega_z/dt$ are the component of the angular acceleration.

Moments and Products of Inertia In terms of a given coordinate system *xyz*, the **inertia matrix** of an object is defined by [Eq. (9.11)]

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \int_m (y^2+z^2)\,dm & -\int_m xy\,dm & -\int_m xz\,dm \\ -\int_m yx\,dm & \int_m (x^2+z^2)\,dm & -\int_m yz\,dm \\ -\int_m zx\,dm & -\int_m zy\,dm & \int_m (x^2+y^2)\,dm \end{bmatrix}$$

where *x, y*, and *z* are the coordinates of the differential element of mass *dm*. The terms I_{xx} , I_{yy} , and I_{zz} are the moments of inertia about the *x, y*, and *z* axes, and I_{xy} , I_{yz} and I_{zx} are the **products of inertia**